Two paths to finding the pitchfork bifurcation in motivation dynamics

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Abstract— **We perform two different model reduction techniques on a dynamical system recently used to develop mechanisms for switching between low-level control vector fields. Both techniques result in the same reduced model which is shown to be a one-parameter unfolding of the supercritical pitchfork bifurcation. The first technique uses singular perturbation to reduce the original two-dimensional system to a one-dimensional system whose vector field is a rational polynomial. The second technique uses the Lyapunov-Schmidt reduction to find another one-dimensional system. In a singular limit analogous to that of the first technique, the Lyapunov-Schmidt reduced dynamics are identical to the rational polynomial found directly through singular perturbation. A nonlinear time scaling argument then shows that the rational polynomial is equivalent to a normal form for the unfolding of the pitchfork.**

I. Introduction

A common method for developing complex control systems is to take a hierarchical approach: encode various low-level objectives in individual control vector fields and then construct a mechanism for the system to switch among the low-level control vector fields as required to perform complex behaviors. The standard switching mechanism used in the literature is the finite-state automaton, which results in a control system that can be modeled as a hybrid dynamical system [1]. This work is part of a broader project that aims to develop an alternative switching mechanism based on continuous dynamical systems. Such a continuous mechanism would present several advantages, including the ability to express a richer set of decisions than a discrete system could and an internal structure that naturally interfaces with the dynamical structure of a mechanical control system [2].

Our framework for developing continuous switching mechanisms, which we term *motivation dynamics*, was first published in [3]. In that paper we developed a system that applies a control input consisting of a convex combination of N low-level control vector fields F_i. The weights, which we term the *motivation* state of the system, are endowed with dynamics modified from a bio-inspired decision-making model published in [4]. The system maintains a state representing the current value of following each of the N control vector fields, termed the *value* state, and the motivation state dynamics is designed to put weight on the task that is most valuable at any given time. In [3], we develop dynamics for the value state and derive sufficient conditions such that, for a particular choice of control vector fields F_1, F_2, the overall control framework exhibits a Hopf bifurcation producing a limit cycle in which the system follows each of the two control vector fields in order.

The motivation dynamics presented in [3] consists of a complex nonlinear vector field. While it is well understood that these dynamics exhibit a pitchfork bifurcation [5], the details of how this pitchfork bifurcation is realized remain poorly studied. In this paper we show that the pitchfork bifurcation arises from a one-parameter unfolding of a symmetric pitchfork singularity. In another recent paper [6], we show that a one-parameter unfolding of the pitchfork analogous to the one found in this paper can generate the Hopf bifurcation studied in [3]. Taken together, this paper and [6] elucidate the dynamical mechanisms at play in [3]. The insight they provide will prove useful in generalizing the original motivation dynamics result to account for general control vector fields.

The remainder of the paper is structured as follows. In Section II we present the motivation dynamics system and recall relevant background on the unfolding theory of bifurcations. In Section III we carry out a reduction of the motivation dynamics system using singular perturbation and show that the resulting reduced system is equivalent to a one-parameter unfolding of the pitchfork. In Section IV we carry out a second reduction of the motivation dynamics system using the Lyapunov-Schmidt reduction and show that the system indeed exhibits a one-parameter unfolding of the symmetric pitchfork bifurcation. We take a singular limit of the Lyapunov-Schmidt-reduced system and show that it yields the system found directly through singular perturbation. Finally, in Section V we interpret these results and conclude.

II. Dynamical systems preliminaries

In this section, we recall the motivation dynamics system studied in [3] and review standard results from the singularity theory of bifurcations concerning the pitchfork bifurcation.

A. Motivation dynamics system

The control system studied in [3] consists of three interconnected dynamical systems which govern the system's physical, motivation, and value states, respectively. The physical state dynamics are taken to be single-integrator dynamics whose input is a combination of N control vector fields:

$$\dot{x} = u := \sum_{i=1}^{N} m_i F_i(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $m_i \in [0, 1]$ is the weight on field i.

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The set of coefficients m_i are referred to as the *motivation* state of the system. The vector m is taken to be an element of the N simplex Δ^N defined by

$$\Delta^N = \left\{ x \in \mathbb{R}^{N+1} \,\middle|\, \sum_{i=1}^{N+1} x_i = 1 \right\}.$$

The interpretation is that m_i represents the non-negative motivation to follow control vector field i, m_{N+1} represents the motivation to not follow any control vector field, and the normalization condition ensures that total motivation remains constant so that the control applied in (1) remains bounded. The dynamics of m depends on a value state defined as follows.

Each control vector field F_i is associated with a state variable $v_i > 0$ that encodes the system's current perception about the value of following that control vector field. The set of v_i states is referred to as the *value* state $v \in \mathbb{R}^N_+$ of the system. In [3], the value state is endowed with dynamics that couple the value of each v_i to the value of a Lyapunov function for the associated control vector field F_i. The specific form of these dynamics is not important for the results presented in this paper.

The work in this paper focuses on the dynamics of the motivation state m. In [3], the authors focus on the case of $N = 2$ control vector fields and endow m with dynamics adapted from [4]. The dynamics of state $m_i, i \in \{1, 2\}$ are given by

$$\dot{m}_i = K v_i m_3 - m_i(1/K v_i - K v_i m_3 + \sigma(1 - m_i - m_3)),$$

where $\sigma > 0$, $v \in \mathbb{R}^2_+$, and $K > 0$ is a positive gain constant. The dynamics of m_3 follow from the constraint $m_1 + m_2 + m_3 = 1$ that defines the simplex. These dynamics can be written compactly as

$$\dot{m} = f_m(m; \sigma, Kv), \tag{2}$$

where $m \in \Delta^2$. It is well understood [4], [5] that these dynamics exhibit a pitchfork bifurcation under appropriate conditions, and intuitively, the motivation dynamics system uses the bifurcation to ensure that the high-value control vector field receives the majority of the motivation weight. The bifurcation is more readily analyzed by expressing $m \in \Delta^2$ in terms of mean-difference coordinates defined by

$$\Delta m = m_1 - m_2, \bar{m} = \frac{m_1 + m_2}{2}$$

and likewise for $v \in \mathbb{R}^2_+$ by defining Δv and $\bar{v}$ in the analogous way. Note that the definitions of these new coordinates and the definitions of Δ^2 and $\mathbb{R}^2_+$ imply that $\bar{m}, \bar{v} > 0$ and that $-2\bar{m} \le \Delta m \le 2\bar{m}$ and $-2\bar{v} < \Delta v < 2\bar{v}$.

In the mean-difference coordinates, the dynamics (2) of $m = (\Delta m, \bar{m})$ take the form

$$\begin{aligned}\dot{\Delta m} &= f_{\Delta m}(\Delta m, \bar{m}; \bar{v}; \Delta v) \\ &= -\left(\frac{2\bar{m} + \Delta m}{K(2\bar{v} + \Delta v)} - \frac{2\bar{m} - \Delta m}{K(2\bar{v} - \Delta v)}\right) \\ &\quad + K\bar{v}\Delta m(1 - 2\bar{m}) + K\Delta v(1 - 2\bar{m})(1 + \bar{m}),\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{\bar{m}} =& f_{\bar{m}}(\Delta m, \bar{m}; \bar{v}, \sigma; \Delta v) \\ =& \frac{1}{2}\Big(-\frac{2\bar{m} + \Delta m}{K(2\bar{v} + \Delta v)} - \frac{2\bar{m} - \Delta m}{K(2\bar{v} - \Delta v)} \\ &+ \frac{K(2\bar{v} + \Delta v)}{2}(1 - 2\bar{m})(1 + \frac{2\bar{m} + \Delta m}{2}) \\ &+ \frac{K(2\bar{v} - \Delta v)}{2}(1 - 2\bar{m})(1 + \frac{2\bar{m} - \Delta m}{2}) \\ &- \frac{\sigma}{2}(2\bar{m} + \Delta m)(2\bar{m} - \Delta m)\Big).\end{aligned} \tag{4}$$

These are the dynamics we will study in detail in the remainder of the paper.

B. Unfolding of a pitchfork bifurcation

The dynamics (2) exhibit a pitchfork bifurcation [4], [5]. As shown in Theorem 3 below, when $\Delta v = 0$, i.e., when $v_1 = v_2$, the bifurcation is symmetric in the sense that it inherits an odd symmetry in Δm from (3). When $\Delta v \neq 0$, the symmetry is broken. We are interested in understanding the mechanism of this symmetry breaking.

A standard way to develop such an understanding is through the so-called *singularity theory* of bifurcations, which is a general theory of the qualitative structural properties of bifurcation problems. We refer the reader to [7] for further information about the singularity theory of bifurcations. In the singularity theory approach, one calls a function F a *perturbation* of a bifurcation problem $f(x, \mu) = 0$ if F is a function $F(x, \mu, \alpha_1, \ldots, \alpha_k)$ forming a k-parameter family of bifurcation problems such that $F(x, \mu, 0, \ldots, 0) = f(x, \mu)$. For many bifurcation problems f, a special perturbation F exists such that any perturbation of f whatsoever is (topologically) equivalent to $F(\cdot, \cdot, \alpha)$ for some $\alpha \in \mathbb{R}^k$ near the origin. Such a special perturbation is called a *universal unfolding* of f.

One such example is the supercritical pitchfork bifurcation, whose normal form is given by

$$\dot{x} = f(x, \mu) := x(\mu - x^2), \tag{5}$$

which defines f. This system has the following equilibria:

$$x = \begin{cases} 0, & \forall \mu \\ \pm\sqrt{\mu}, & \mu > 0. \end{cases}$$

The nonzero equilibria are stable when they exist; the equilibrium $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. It is a well-known result [7, I.1.13] that

$$F(x, \mu, \alpha) = x(\mu - x^2) + \alpha_1 + \alpha_2 x^2 \tag{6}$$

is a universal unfolding of the pitchfork (5). In the following, we show that the motivation dynamics (2) embeds a particular unfolding of the pitchfork (5). We show this embedding by carrying out two different reduction procedures on the dynamics (2).

III. Reduction via singular perturbation

The first of the two reductions of the dynamics (2) is via singular perturbation, for which [8] is a standard reference. In [3], the authors studied the motivation dynamics control system using singular perturbation and found conditions such that the coupled system exhibits a limit cycle. In this section, we use singular perturbation on the motivation dynamics (2) in isolation to elucidate the decision mechanism that these dynamics embed. As such, we consider the value state $v = (\Delta v, \bar{v})$ as a set of (unfolding) parameters.

The insight underlying the singular perturbation analysis is that the dynamics (2) simplify significantly in the limit $K \to +\infty$. We formalize this insight by posing (2) in the form of a singular perturbation problem. Let $\epsilon = 1/K, x = \Delta m$, and $y = (1-2\bar{m})/\epsilon$.[1] In the (x, y) coordinates, the dynamics (2) become the system

$$\begin{aligned}
\dot{x} &= f_x(x, y; v, \epsilon) && (7)\\
&= -\epsilon\left(\frac{1-\epsilon y + x}{2\bar{v}+\Delta v} - \frac{1-\epsilon y - x}{2\bar{v}-\Delta v}\right) + \bar{v}xy + \Delta v y\frac{3-\epsilon y}{2}
\end{aligned}$$

$$\begin{aligned}
\epsilon\dot{y} &= g_y(x, y; v, \epsilon) && (8)\\
&= \epsilon\left(\frac{1-\epsilon y + x}{2\bar{v}+\Delta v} - \frac{1-\epsilon y - x}{2\bar{v}-\Delta v}\right) + \frac{\sigma}{2}((1-\epsilon y)^2 - x^2)\\
&\quad - \frac{y}{2}\left((2\bar{v}+\Delta v)\left(1+\frac{1-\epsilon y + x}{2}\right)\right)\\
&\quad - \frac{y}{2}\left((2\bar{v}-\Delta v)\left(1+\frac{1-\epsilon y - x}{2}\right)\right).
\end{aligned}$$

Taking the singular limit yields a reduced system whose dynamics are given by a rational polynomial, as formalized in the following theorem.

Theorem 1: In the singular limit $\epsilon \to 0$, the motivation dynamics (2) reduce to

$$\dot{x} = \frac{\sigma}{2}(1-x^2)\frac{2x+3\alpha}{6+\alpha x}, \tag{9}$$

where $\alpha = \Delta v/\bar{v}$.

Proof: The proof follows the standard procedure for analyzing singularly-perturbed systems. First note that x is the slow and y the fast variable. Taking the singular limit $\epsilon \to 0$ of (7) and (8) yields

$$\dot{x} = f_x(x, y; v, 0) = \bar{v}xy + \frac{3\Delta v y}{2} \tag{10}$$

$$0 = g_y(x, y; v, 0) = -\frac{y}{2}(6\bar{v} + \Delta v x) + \frac{\sigma}{2}\left(1-x^2\right). \tag{11}$$

Solving Equation (11) for the fast variable y yields

$$y = h(x) := \frac{\sigma(1-x^2)}{6\bar{v}+\Delta v x} = \frac{\sigma}{\bar{v}}\frac{1-x^2}{6+(\Delta v/\bar{v})x},$$

which defines the slow manifold $\{(x, y) = (x, h(x))\}$. The system quickly converges to the slow manifold and then x slowly evolves on the slow manifold. Using the expression $y = h(x)$ for the fast variable y in terms of the slow variable x yields the reduced slow dynamics

$$\dot{x} = f_x(x, h(x); v, 0) = \frac{\sigma}{2}(1-x^2)\frac{2x+3\Delta v/\bar{v}}{6+(\Delta v/\bar{v})x}.$$

Defining $\alpha = \Delta v/\bar{v}$ yields the desired result (9). ■

The reduced dynamics (9) are significantly simpler than the original dynamics (2). However, their rational form can be further simplified by a nonlinear state-dependent time scaling, which yields a system that is clearly an unfolded pitchfork of the form (6). This result is formalized in the following corollary.

Corollary 2: The singularly-perturbed motivation dynamics (9) are equivalent to

$$x' = x(1-x^2) + \frac{3}{2}\alpha - \frac{3}{2}\alpha x^2,$$

i.e., an unfolding of the pitchfork bifurcation (6) with bifurcation parameter $\mu \mapsto 1$ and unfolding parameters $\alpha_1 = 3\alpha/2$ and $\alpha_2 = -3\alpha/2$.

Proof: Begin with the reduced slow motivation dynamics (9) and recall the definition $\alpha = \Delta v/\bar{v}$. Note that by the definitions of Δv and $\bar{v}$ we have $-2\bar{v} \leq \Delta v \leq 2\bar{v}$ and therefore that $-2 \leq \alpha \leq 2$. Similarly, by the definition of $x = \Delta m$ we have $-1 \leq x \leq 1$. Next, define $s(x) = (6+\alpha x)/\sigma$. The bounds on α and x imply that s obeys $0 < 4/\sigma \leq s(x) \leq 8/\sigma < \infty$.

Now, simplify the slow dynamics by introducing a nonlinear time scaling as is common practice in the literature [9], [10]. Write

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = s(x).$$

By the bounds on $s(x)$, it is clear that t is a monotonically-increasing function of the new time τ, so τ is a well-defined time variable. Then, since

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

we have

$$x' = \frac{\mathrm{d}x}{\mathrm{d}\tau} = f_x(x, h(x); v, 0)s(x) = (1-x^2)\left(x + \frac{3\alpha}{2}\right). \tag{12}$$

Expanding the product yields

$$x' = x(1-x^2) + \frac{3\alpha}{2} - \frac{3\alpha}{2}x^2. \tag{13}$$

By comparing the system (13) to the universal unfolding of the pitchfork (6), it is clear that the two expressions coincide if $\mu = 1, \alpha_1 = 3\alpha/2$, and $\alpha_2 = -3\alpha/2$. ■

The results in this section show that the motivation dynamics (2) embed dynamics that are equivalent to an unfolded pitchfork bifurcation. In the next section, we perform a second reduction procedure on the dynamics (2) and show that the dynamics are equivalent to a one-parameter unfolding of the symmetric pitchfork bifurcation (5).

[1]Note that this definition, which we make for consistency with [8], overloads the previous definition of x as the control system's physical state.

IV. Lyapunov-Schmidt reduction

In the previous section we showed that the dynamics (2) embed dynamics that are equivalent to an unfolded pitchfork bifurcation. It is well understood [4], [5] that, when $\Delta v = 0$, these dynamics exhibit a pitchfork bifurcation as the parameter σ is raised through a critical value σ^*. In this section, we compute the Lyapunov-Schmidt reduction of the dynamics (2) in a neighborhood of the singular point associated with the previously-known bifurcation. We then take a singular limit of the resulting reduced system and show that it yields the system found in Theorem 1. This shows that the unfolded pitchfork system arises from the previously-known pitchfork bifurcation.

A. Singular point

We begin by finding the singular point of (2) associated with the known pitchfork bifurcation. Note that when $\Delta v = 0$, the component dynamics (3), (4) reduce to

$$\dot{\Delta m} = \Delta m \frac{-1 + (1-2\bar{m})K^2\bar{v}^2}{K\bar{v}}$$

and

$$\dot{\bar{m}} = K\bar{v} - \left(\frac{1}{K\bar{v}} + K\bar{v}\right)\bar{m} - (2K\bar{v}+\sigma)\bar{m}^2 - \frac{\sigma}{4}\Delta m^2.$$

Note that these dynamics have an equilibrium where $m = m_d = (\Delta m_d, \bar{m}_d)$, with $\Delta m_d = 0$ and $\bar{m}_d = \frac{-(1+K^2\bar{v}^2)+\sqrt{1+2K^2\bar{v}^2+4\sigma K^3\bar{v}^3+9K^4\bar{v}^4}}{2K\bar{v}(2K\bar{v}+\sigma)}$. At this equilibrium, the Jacobian of the dynamics (2) evaluates to

$$J_d = \begin{bmatrix} j_{11} & 0 \\ 0 & j_{22} \end{bmatrix},$$

where the components are given by

$$\begin{aligned} j_{11} = -\frac{1}{K\bar{v}(2K\bar{v}+\sigma)}\Big(&\bar{v}(1-3K^2\bar{v}^2) + \sigma(1-K^2\bar{v}^2) \\ &+ K\bar{v}\sqrt{1+2K^2\bar{v}^2+9K^4\bar{v}^4+4\sigma K^3\bar{v}^3}\Big), \end{aligned}$$

$$j_{22} = -\frac{\sqrt{1+2K^2\bar{v}^2+9K^4\bar{v}^4+4\sigma K^3\bar{v}^3}}{K\bar{v}}.$$

In this case with $\Delta v = 0$, the equilibrium $m = m_d$ of (2) is a singularity at the appropriate value of σ and $\bar{v}$. In particular, for $K\bar{v} > 1$, the singular value of σ is

$$\sigma = \sigma^* = \frac{4K^3\bar{v}^3}{(1-K^2\bar{v}^2)^2}. \tag{14}$$

At the singularity, $j_{11} = 0$.

Thus, the dynamics (2) have a singularity when $(\Delta m, \bar{m}; \Delta v, \bar{v}, \sigma, K) = (0, \bar{m}_d, 0, \bar{v}, \sigma^*(K,\bar{v}), K)$. When $\Delta v = 0$, the pitchfork bifurcation is symmetric; when $\Delta v \neq 0$, the symmetry is broken and the bifurcation unfolds in an asymmetric manner. We wish to understand that the structure of the unfolding.

B. Reduction at the singular point

We now compute the Lyapunov-Schmidt reduction of the dynamics (2) at the singular point. Effectively, the Lyapunov-Schmidt reduction consists of the projection of the dynamics into the kernel of the Jacobian at the singular point. See Golubitsky and Schaeffer [7]. We proceed by performing the Lyapunov-Schmidt reduction as described in [7, Chapter I.3]. Our notation follows that of Golubitsky and Schaeffer.

We identify a neighborhood of $m_d \in \Delta^2$ with $\mathbb{R}^2$ and proceed with the reduction in this neighborhood. Let $L = J_d(\sigma = \sigma^*)$ be the Jacobian of the (2) evaluated at the critical value $\sigma = \sigma^*$ established in (14). Note that $\ker L$ is the space $\left\{\begin{bmatrix} x_1 & 0\end{bmatrix}^T : x_1 \in \mathbb{R}\right\}$ and that $\operatorname{range} L$ is the space $\left\{\begin{bmatrix} 0 & x_2\end{bmatrix}^T : x_2 \in \mathbb{R}\right\}$. Furthermore, note that the kernel and the range of L split $\mathbb{R}^2$: $\ker L \oplus \operatorname{range} L = \mathbb{R}^2$. We select vector space complements to $\ker L$ and $\operatorname{range} L$ equal to $M = \operatorname{range} L$ and $N = \ker L$, respectively. Then

$$\mathbb{R}^2 = \ker L \oplus M \text{ and } \mathbb{R}^2 = N \oplus \operatorname{range} L. \tag{15}$$

Having defined the splitting (15) we let $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and note that E is the projection of $\mathbb{R}^2$ onto $\operatorname{range} L$ and that $\ker E = N$. The complementary projection $I - E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has range equal to N and kernel equal to $\operatorname{range} L$.

An equilibrium $m_e \in \Delta^2$ of (2) solves $f_m(m; \sigma, v) = 0$. This equation can be expanded into the equivalent system of two equations

$$\begin{aligned} Ef_m(m;\sigma,v) &= 0 \\ (I-E)f_m(m;\sigma,v) &= 0 \end{aligned}$$

using the two complementary projections. Note that the first equation corresponds to $\dot{\bar{m}} = 0$ and the second to $\dot{\Delta m} = 0$. Let $m = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = \begin{bmatrix} \Delta m & 0 \end{bmatrix}^T \in \ker L$ and $\mathbf{w} = \begin{bmatrix} 0 & \bar{m} \end{bmatrix}^T \in M$.

Near the equilibrium point m_d, the implicit function theorem guarantees that the equation $\dot{\bar{m}} = 0$ can be solved for $\bar{m}$. In fact, as a quadratic equation in $\bar{m}$, it can be solved exactly, yielding the one relevant solution $\bar{m} = \bar{m}^*(\Delta m, \Delta v, \bar{v}, K, \sigma)$ given by

$$\bar{m}^*(\Delta m, \Delta v, \bar{v}, K, \sigma) = \frac{a + \sqrt{b}}{c}, \tag{16}$$

where a, b, and c are defined as

$$\begin{aligned} a = {} & K^3\Delta m\Delta v^3 - 8K\bar{v} + 2K^3\Delta v^2\bar{v} \\ & - 4K^3\Delta m\Delta v\bar{v}^2 - 8K^3\bar{v}^3 \end{aligned} \tag{17}$$

$$\begin{aligned} b = {} & \big[\Delta m K^3(\Delta v^3 - 4\Delta v\bar{v}^2) \\ & \quad +2K\bar{v}(K^2\Delta v^2 - 4(1+K^2\bar{v}^2))\big] \\ & + 4K^5(\Delta v^2 - 4\bar{v}^2)(2K\bar{v}+\sigma)(4\bar{v}(\Delta v^2 - 4\bar{v}^2) \\ & + \Delta m K\Delta v(K^2\Delta v^2 - 4(1+K^2\bar{v}^2)) \\ & + \Delta m^2K^2(\Delta v^2 - 4\bar{v}^2)\sigma) \end{aligned} \tag{18}$$

$$c = 4K^2(4\bar{v}^2 - \Delta v^2)(2K\bar{v}+\sigma). \tag{19}$$

Note that $\bar{m}^*$ is invariant under the transformation $(\Delta m, \Delta v) \mapsto (-\Delta m, -\Delta v)$.

The Lyapunov-Schmidt reduction of (2) at the singular point, then, comes from expressing the second projected equation in coordinates after substituting in the value of $\bar{m}$ defined by (16). As in Theorem 1, let $\alpha = \Delta v/\bar{v}$. This yields

$$\begin{aligned}\dot{\Delta m} &= g(\Delta m, \bar{v}, \sigma, \alpha, K) \\ &:= f_{\Delta m}(\Delta m, \bar{m}^*(\Delta m, \alpha\bar{v}, \bar{v}, K, \sigma); \bar{v}, \sigma, \alpha\bar{v}),\end{aligned} \tag{20}$$

which defines the reduced function g. It is straightforward to see that $f_{\Delta m}$ satisfies

$$f_{\Delta m}(\Delta m, \bar{m}; \bar{v}, \sigma, \Delta v) = -f_{\Delta m}(-\Delta m, \bar{m}; \bar{v}, \sigma, -\Delta v),$$

which implies that the analogous symmetry holds for g, namely, that

$$g(\Delta m, \bar{v}, \sigma, \alpha, K) = -g(-\Delta m, \bar{v}, \sigma, -\alpha, K).$$

In particular, when $\Delta v = 0$, g is an odd function of Δm.

We can now use the reduced function to study the bifurcation properties of the original dynamics (2). The results are summarized in Theorem 3 below.

Let K^* be a solution of $\sigma = 4K^3\bar{v}^3/(-1+K^2\bar{v}^2)^2$ such that $K^*\bar{v} > 1$. Let g defined by (20) be the Lyapunov-Schmidt reduction of the dynamics (2) at $(x, \bar{v}, \sigma, \alpha, K) = (0, \bar{v}, \sigma, 0, K^*)$.

Theorem 3: The following statements hold for the Lyapunov-Schmidt reduction (20) of the motivation dynamics (2):

i) The bifurcation problem $g(x, \bar{v}, \sigma, 0, K)$ has a symmetric pitchfork singularity at $(x, K) = (0, K^*)$.
ii) For $\alpha \neq 0$, the bifurcation problem $g(x, \bar{v}, \sigma, \alpha, K)$ is a one-parameter unfolding of the symmetric pitchfork.

Proof: We prove the two statements in order. By [7, Proposition II.9.2], computing several derivatives of the reduced function g at the singular point is sufficient to prove the first statement. By symmetry, we have that $g(0, \bar{v}, \sigma, 0, K) = 0$, which restates the fact that $\Delta m = 0$ is an equilibrium when $\Delta v = 0$. Straightforward but tedious calculations show that

$$\frac{\partial g}{\partial x}(0, \bar{v}, \sigma, 0, K^*) = 0.$$

Similarly, we have

$$\frac{\partial g}{\partial K}(0, \bar{v}, \sigma, 0, K^*) = \frac{\partial^2 g}{\partial x^2}(0, \bar{v}, \sigma, 0, K^*) = 0,$$

and that

$$\frac{\partial^3 g}{\partial x^3}(0, \bar{v}, \sigma, 0, K^*) = -\frac{12(K^*\bar{v})^5}{((K^*\bar{v})^2-1)(1+3(K^*\bar{v})^4)} < 0$$

$$\frac{\partial^2 g}{\partial x \partial K}(0, \bar{v}, \sigma, 0, K^*) = \frac{2\bar{v}(3+(K^*\bar{v})^2)}{1+3(K^*\bar{v})^4} > 0,$$

where the two inequalities follow from the assumption that $K^*\bar{v} > 1$. The first statement then follows from applying [7, Proposition II.9.2].

The second statement is just the definition of a one-parameter unfolding. ■

C. Singular limit of the Lyapunov-Schmidt reduction

The result stated in Theorem 3 confirms that the bifurcation exhibited by the motivation dynamics (2) is a one-parameter unfolding of the symmetric pitchfork bifurcation and that $\alpha = \Delta v/\bar{v}$ is the relevant unfolding parameter. However, this result relies on the Lyapunov-Schmidt reduction and therefore is, in principle, only valid in a neighborhood of the singular point $(\Delta m, K) = (0, K^*)$. In fact, the reduction converges to a well-defined function in the singular limit $K \to \infty$ (i.e., $\epsilon \to 0$). Furthermore, this limiting function is identical to the slow dynamics (9) found in Theorem 1. This results are made precise in Theorem 4 below. Taken together, Theorems 3 and 4 establish the mechanism by which the dynamics (2) undergo bifurcation and result in the slow system (9).

Theorem 4: In the limit $K \to \infty$, the Lyapunov-Schmidt reduction (20) reduces to

$$g_r(x, \bar{v}, \sigma, \alpha) = \frac{\sigma}{2}(1-x^2)\frac{2x+3\alpha}{6+\alpha x}. \tag{21}$$

This is identical to the slow dynamics (9) from Theorem 1.

Proof: It suffices to show that the limit $\lim_{K\to\infty} g(x, \bar{v}, \sigma, K, \alpha)$ is well defined and equal to $g_r(x, \bar{v}, \sigma, \alpha)$. First, note that the relevant dynamics (20) can be rewritten using (3) as

$$\begin{aligned}\dot{x} &= g(\Delta m, \bar{v}, \sigma, \alpha, K) = f_{\Delta m}(x, \bar{m}^*; \bar{v}, \sigma; \alpha\bar{v}) \\ &= -\left(\frac{2\bar{m}^*+x}{K\bar{v}(2+\alpha)} - \frac{2\bar{m}^*-x}{K\bar{v}(2-\alpha)}\right) \\ &\quad + K\bar{v}x(1-2\bar{m}^*) + K\alpha\bar{v}(1-2\bar{m}^*)(1+\bar{m}^*),\end{aligned} \tag{22}$$

where $\bar{m}^*$ is given by (16).

Straightforward computation shows that

$$\lim_{K\to\infty} \bar{m}^*(x, \alpha\bar{v}, \bar{v}, K, \sigma) = \frac{1}{2}$$

and that

$$\lim_{K\to\infty} K\bar{v}(1-2\bar{m}^*(x, \alpha\bar{v}, \bar{v}, K, \sigma)) = \sigma\frac{1-x^2}{6+\alpha x}.$$

The interpretation behind the first limit is that, in the limit $K \to \infty$, the motivation state is fully committed, i.e., $m_{N+1} = 0$. The second limit essentially recovers the slow manifold derived from (11).

Note that $-2\bar{v} < \Delta v < 2\bar{v}$ implies that $-2 < \alpha < 2$. Thus, it is clear that the limit $\lim_{K\to\infty} g(x, \bar{v}, \sigma, K, \alpha)$ is well defined. Taking the limit of g and using the two limits computed above yields

$$g_r(x, \bar{v}, \sigma, \alpha) = \left(x + \frac{3\alpha}{2}\right)\sigma\frac{1-x^2}{6+\alpha x},$$

which, upon rearranging, yields the desired result (21). The comparison to (9) is obvious. ■

The following corollary is analogous to Corollary 2 and its proof is identical to that of Corollary 2.

Corollary 5: The dynamics $\dot{x} = g_r(x, \bar{v}, \sigma, \alpha)$ defined in (21) are equivalent to an unfolding of the pitchfork bifurcation (6) with unfolding parameters $\alpha_1 = 3\alpha/2$ and $\alpha_2 = -3\alpha/2$.

x	stable for
$+1$	$\alpha > -2/3$
-1	$\alpha < 2/3$
$-\frac{3\alpha}{2}$	$-2/3 < \alpha < 2/3$

TABLE I

EQUILIBRIA AND STABILITY PROPERTIES OF THE SINGULARLY REDUCED DYNAMICS (9).

As stated in the introduction to this section, the implication of these results is to show how the dynamics (2) undergo an unfolded pitchfork bifurcation to yield the dynamics (9) in the singular limit. The limit calculation in Theorem 4 makes it clear that it is the term $xK\bar{v}(1-2\bar{m}^*)$ that embeds the symmetric pitchfork and the term $\alpha K\bar{v}(1+\bar{m}^*)(1-2\bar{m}^*)$ that embeds the unfolding.

The equilibria and associated stability properties of the reduced dynamics (9) are given in Table I and shown visually in Figure 1. Note that, due to the definition of the simplex Δ^2, equilibrium values x are only realizable if they are in the interval $[-1, 1]$. The intermediate equilibrium $-3\alpha/2$ is only realizable for $\alpha \in [-2/3, 2/3]$, which corresponds to the values of α where the intermediate equilibrium is stable.

Understanding of this mechanism provides additional insight into the behavior of the motivation dynamics system studied in [3]. For example, examination of Figure 1 makes it clear that the motivation system switches from $\Delta m = +1$ (corresponding to putting all the weight on control vector field F_1) to $\Delta m = -1$ (corresponding to putting all the weight on control vector field F_2) when the unfolding parameter α decreases below $-2/3$. The value $2/3$ arises from the unfolding parameter $3\alpha/2$, which itself comes from the limiting value of $(1+\bar{m}^*)$ in the reduced dynamics (22). By changing the $(1+\bar{m}^*)$ term to $(1+\beta\bar{m}^*)$, $\beta \in \mathbb{R}$, we can control the value of α at which the system switches tasks.

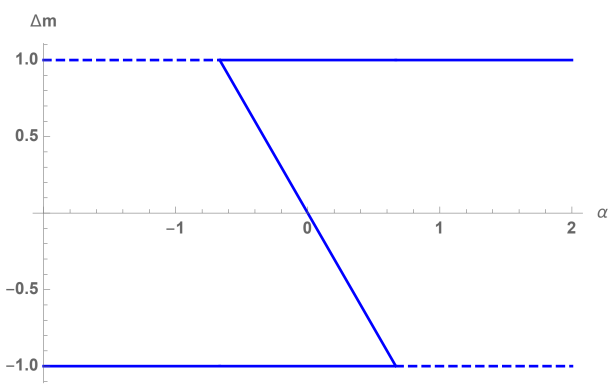

Fig. 1. Equilibria of the singularly-reduced dynamics (9) as a function of the unfolding parameter α. Solid lines represent stable equilibria; dashed lines unstable equlibria.

V. CONCLUSION

In this paper, we considered the control system introduced in [3] and studied the core decision mechanism in detail. The decision mechanism, whose dynamics are given by (2), was understood to embed a pitchfork bifurcation, but the details of this embedding were not well studied.

In this work, we performed two separate reductions to understand the details of the embedding. We first carried out a reduction using singular perturbation theory and showed that the singularly-perturbed system resulted in a set of slow reduced dynamics which are equivalent to a particular unfolding of the symmetric pitchfork bifurcation. We then carried out a second reduction in the neighborhood of the previously-known singular (i.e., bifurcation) point of the dynamics using the Lyapunov-Schmidt formalism. Using this second reduction, we formally showed the existence of a one parameter unfolding of the pitchfork in the original dynamics (2). Finally, we took a singular limit of the dynamics found in the second reduction and showed that, in this limit, the reduced dynamics are identical to those found directly through singular perturbation. Taken together, these results thoroughly characterize the bifurcation properties of the motivation dynamics (2).

There are several implications of these results. First, they are of independent technical interest due to the wide range of application of such bifurcation mechanisms in a variety of scientific domains, e.g., [4], [5]. Secondly, they are useful for understanding how to construct bifurcation mechanisms in order to achieve desired control outcomes. Bifurcation control is a longstanding area in its own right [11], and has been used in a number of recent works, e.g., [12]. For our own work, we intend to use the insight provided by the analysis in this paper to further develop control systems based on motivation dynamics. The insight from Table I will allow better control over the thresholding and other effects inherent in the dynamics (2).

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